COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA 17 (4) (1997) 447–458

ON LINEAR COMBINATORICS I. CONCURRENCY — AN ALGEBRAIC APPROACH

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Received July 26, 1996
Revised January 29, 1998

Dedicated to the memory of P. Erdős

This article is the first one in a series of three. It contains concurrency results for sets of linear mappings of ℝ with few compositions and/or small image sets. The fine structure of such sets of mappings will be described in part II [3]. Those structure theorems can be considered as a first attempt to find Freiman–Ruzsa type results for a non-Abelian group. Part III [4] contains some geometric applications.

0. Introduction

Combinatorial problems on incidences of point sets and straight lines have been studied since Jackson [9] and Sylvester [13, 14, 15, 16]. Much later Gallai found the first combinatorial distinction between Euclidean and finite geometries in [7], see also Beck [2] and Szemerédi–Trotter [18].

The goal of this paper (actually the goal of the sequence of articles of which this one is the first member) is to prove structure theorems specific to Euclidean geometry — i.e., some which do not hold true within finite planes.

In this Part I, an algebraic approach is taken. We study straight lines as linear real functions and find weak structures under various assumptions — typically large concurrent or parallel subsets. (As usual, it is the hard part to find *any* structure; the *deeper* structure comes easier then.)

In part II [3], the results of this part I are utilized and stronger structure theorems are deduced — generalizing some results of Freiman and Ruzsa to the non-Abelian group of linear functions with composition as the group operation.

Finally, part III [4] is pure geometry. Distorted grids, incidences and sets which determine few directions are mentioned there.

Mathematics Subject Classification (1991): 52C10, 51A25, 20F12

* Research partially supported by HU-NSF grants OTKA T014105, T014302 and T019367.

1. Composition sets and concurrency.

Throughout this paper $\mathcal{L}$ will denote the set of non-constant linear functions $x \mapsto ax+b$ $(a \neq 0)$, i.e., the non-degenerate affine mappings of $\mathbb{R}$.

We shall use the following (probably standard) notation: if $\Box$ is an arbitrary operation of two variables, and A, B are arbitrary sets, then we put

$$A\Box B = \{a\Box b\ ;\ a \in A\ ,\ b \in B\}.$$

Finite sets of linear mappings will usually be denoted by Φ or Ψ; the operation of our main concern is the composition $\phi \circ \psi$.

The problem we address is the following: what is the structure of Φ and Ψ if $\Psi \circ \Phi$ is not too big as compared to Φ and Ψ?

Question 1. What structure can Φ and Ψ have if $\Phi, \Psi \subset \mathcal{L}$, $|\Phi| = |\Psi| = n$ and, say,

$$|\Psi \circ \Phi| \leq 1000n?$$

(Of course, any positive constant C could be considered in place of 1000 above.)

In what follows, we call $\Psi \circ \Phi$ a *composition set.*

The motivations to this problem are twofold. On the one hand, some purely geometric investigations (see part III [4]) require the solution to problems like this. On the other hand, Question 1 is a close relative of some results in combinatorial number theory. We give some details of the latter aspect first.

Freiman [5, 6] and Ruzsa [11, 12] studied subsets of $\mathbb{R}$, for which

$$|A+B| \leq Cn$$

where $|A| = |B| = n$.

They described the structure of A and B in terms of some natural generalizations of arithmetic progressions (see also part II [3] for more details). Their results extend to any torsion-free Abelian group.

However, the operation of composing two mappings does not commute in general. That is why we shall rather consider two, essentially different types of composition sets:

(a) asymmetric ones (like those above); and
(b) symmetric ones, like $\Phi \circ \Psi \cup \Phi \circ \Psi$.

Question 2. (Symmetric composition sets) What is the structure of Φ and Ψ if

$$|\Phi \circ \Psi \cup \Phi \circ \Psi| \leq 1000n$$

for some $\Phi, \Psi \subset \mathcal{L}$ with $|\Phi| = |\Psi| = n$?

It is worth to note that the symmetric assumption of Question 2 is, indeed, stronger than that of Question 1, as demonstrated by some examples in part II [3].

As a matter of fact, Questions 1–2 can be considered as attempts to finding Freiman-Ruzsa type structure theorems for the non-Abelian group $\mathscr{L}$ with operation "$\circ$".

The goal of this first part is to find a certain (weak) structure; i.e., we shall show here that many of the Φ and Ψ must be concurrent or parallel. Their finer structure will be described later on in part II [3].

Theorem 1. *For every $C>0$ there is a $c^* = c^*(C)>0$ with the following property. Assume that*

$$|\Phi \circ \Psi| \leq Cn$$

for some $\Phi, \Psi \subset \mathscr{L}$ with $|\Phi|, |\Psi| \geq n$. Then Φ and Ψ contain some $\Phi^* \subset \Phi$ and $\Psi^* \subset \Psi$ with $|\Phi^*|, |\Psi^*| \geq c^* n$, for which

(i) *either both Φ^* and Ψ^* consist of parallel lines;*

(ii) *or both Φ^* and Ψ^* consist of concurrent lines.*

2. A symmetric statistical lemma

Actually, we shall prove a so-called "statistical" generalization of Theorem 1 (see Theorem 3 in Section 4). The notion of "statistical" results originates from Balogh and Szemerédi [1], who extended the theorems of Freiman and Ruzsa [6, 12] by relaxing the assumption that *all* pairwise sums must be taken into account, and just considered the sums of *some* cn^2 pairs (a,b). Later on Laczkovich and Ruzsa [10] proved an even stronger statement of this type.

2.1. Statistical composition sets.

Definition 1. For $E \subset A \times B$ and any operation $\Box$, define

$$A \Box_E B = \{a \Box b \ ; \ (a,b) \in E\}.$$

Also, with a small abuse of notation, we will write $B \Box_E A$ for $\{a \Box b \ ; \ (a,b) \in E\}$.

Lemma 2. (Main Lemma) *For every $C>0$ there is a $c^* = c^*(C)>0$ with the following property. Let $\Phi, \Psi \subset \mathscr{L}$ with $n \leq |\Phi|, |\Psi| \leq Cn$ and $E \subset \Phi \times \Psi$ with $|E| \geq n^2$. Assume, moreover, that*

$$|\Phi \circ_E \Psi \cup \Phi \circ_E \Psi| \leq Cn.$$

Then there exist $\Phi^* \subset \Phi$ and $\Psi^* \subset \Psi$ with

$$|(\Phi^* \times \Psi^*) \cap E| \geq c^* n^2, \tag{1}$$

for which

(i) *either both* Φ^* *and* Ψ^* *consist of parallel bundles;*
(ii) *or both* Φ^* *and* Ψ^* *consist of concurrent bundles (with common points* (a,b), *and* (b,a), *respectively).*

Remark 3. Note that (1) implies

$$|\Phi^*|, |\Psi^*| \geq \frac{c^*}{C} \cdot n = c^{**}n.$$

The proof of the above Main Lemma can be found in Section 2.3.

2.2. Commutator pairs and commutator graphs.

Since we are studying a non-Abelian group (i.e., $\mathscr{L}$), it is quite natural to define some notions that can be considered as relatives of the usual commutators.

Definition 4. For $\phi, \psi \in \mathscr{L}$, the pair $(\phi \circ \psi, \psi \circ \phi)$ is called the *commutator pair* defined by ϕ and ψ.

Remark 5. Of course, the two terms of a commutator pair are identical if (and only if) ϕ and ψ commute.

Definition 6. For $\Phi, \Psi \subset \mathscr{L}$ and $E \subset \Phi \times \Psi$, the *commutator graph* $\hat{G}_E(\hat{V}, \hat{E})$ defined by E has edge set $\hat{E}$ which consists of the corresponding commutator pairs, i.e.,

$$\hat{V} = \Phi \circ_E \Psi \cup \Phi \circ_E \Psi; \text{ and}$$
$$\hat{E} = \{(\phi \circ \psi, \psi \circ \phi) \ ; \ (\phi, \psi) \in E\}.$$

Remark 7. Though E may be considered as a directed graph on $\Phi \cup \Psi$, the edge set $\hat{E}$ of the commutator graph will always be undirected.

Remark 8. Again, it is worth to note that an edge $(\phi \circ \psi, \psi \circ \phi)$ is a self-loop iff ϕ and ψ commute.

The following lemma will imply that the Main Lemma (Lemma 2) is true under the additional assumption that the intersection points of pairs of lines (ϕ, ψ^{-1}) for which $(\phi, \psi) \in E$, are all distinct (including that $\phi \neq \psi^{-1}$ for these pairs).

Lemma 9. (Commutator Lemma) *For every* $C > 0$ *there is a* $c^* = c^*(C) > 0$ *with the following property. Let* $\Phi, \Psi \subset \mathscr{L}$ *with* $n \leq |\Phi|, |\Psi| \leq Cn$ *and* $E \subset \Phi \times \Psi$ *with* $|E| \geq n^2$. *Also assume that* $|\Phi \circ_E \Psi \cup \Phi \circ_E \Psi| \leq Cn$. *If, moreover, the intersection points*

$$\Phi \bigcap\nolimits_E \Psi^{-1} = \{\phi \cap \psi^{-1} \ ; \ (\phi, \psi) \in E\}$$

are all distinct (including that $\phi \neq \psi^{-1}$ for $(\phi,\psi) \in E$), then there exists an $E^* \subset E$ with $|E^*| \geq c^* n^2$ for which

(i) *the graph $(\Phi \cup \Psi, E^*)$ consists of one non-empty connected component (leaving, perhaps, some points of $\Phi \cup \Psi$ isolated); and*

(ii) *the commutator subgraph $\hat{G}^*_{E^*}(\hat{V}^*, \hat{E}^*)$ defined by E^* is contained in one connected component of the original "big" commutator graph $\hat{G}_E(\hat{V}, \hat{E})$ defined by E.*

Throughout the proof we shall use the following simple observation a couple of times.

Proposition 10. (Folklore) *Let $0 < c_0 < 1/2$. If an undirected graph with at most N vertices has at least $c_0 N^2$ distinct edges, then it has a connected component with at least $(c_0^2/2)N^2$ edges.* ∎

Proof of the Commutator Lemma (Lemma 9). Without loss of generality we can assume $C > 2$.

1. First we observe that if the intersection points $\Phi \bigcap_E \Psi^{-1}$ are all distinct, then $\hat{G}_E(\hat{V}, \hat{E})$ contains no multiple edges.
   Indeed, if u is the fixed point of $\psi_j \circ \phi_i$ while v is the fixed point of $\phi_i \circ \psi_j$, then $\phi_i \cap \psi_j^{-1} = (u,v)$. These pairs are, therefore, all distinct.
2. Using Proposition 10 for $\hat{E}$, $N = Cn$ and $c_0 = 1/C^2$, we get a connected $H_1(\hat{V}', \hat{E}') \subset \hat{G}$ such that

$$\left|\hat{E}'\right| \geq \frac{1}{2C^4} N^2 = \frac{1}{2C^2} n^2.$$

3. Define $H_2(\Phi \cup \Psi, F)$ to be the (bipartite) graph with edge set

$$F = \{(\phi,\psi) \in E \ ; (\phi \circ \psi, \psi \circ \phi) \in \hat{E}'.\}$$

4. Apply Proposition 10 again to $N = 2Cn$ and $c_0 = 1/(8C^4)$ and get the desired $E^* \subset F$ with $|E^*| \geq \frac{N^2}{2(8C^4)^2} = \frac{n^2}{32C^6} = c^* n^2$.

2.3. Proof of Lemma 2

First, as a consequence of the Commutator Lemma (Lemma 9), we show that if a symmetric composition set is small, then the lines in Φ can only have many distinct intersection points with those in Ψ^{-1} if many of them are parallel. Then, using a theorem of Beck, the proof of Lemma 2 can be completed easily.

Lemma 11. *For every $C>0$ there is a $c^*=c^*(C)>0$ with the following property. Let $\Phi,\Psi\subset\mathcal{L}$ with $n\le|\Phi|,|\Psi|\le Cn$, $E\subset\Phi\times\Psi$ with $|E|\ge n^2$ and*

$$|\Phi\circ_E\Psi\cup\Phi\circ_E\Psi|\le Cn.$$

Assume, moreover, that the intersection points $\Phi\bigcap_E\Psi^{-1}$ are all distinct.

Then there exist $\Phi^*\subset\Phi$ and $\Psi^*\subset\Psi$ with $|(\Phi^*\times\Psi^*)\cap E|\ge c^*n^2$ such that both Φ^* and Ψ^* consist of parallel lines.

Proof. Use the Commutator Lemma (Lemma 9) and denote by Φ^* and Ψ^* the endpoints of the edges of E^* in Φ and Ψ, respectively. We show that both Φ^* and Ψ^* consist of parallel lines.

First observe that the leading coefficients (i.e., slopes) of a commutator pair $(\phi\circ\psi,\psi\circ\phi)$ are equal. Therefore *all* linear functions represented by the vertices of a *connected component* of the commutator graph have identical slopes. As $\hat{G}^*_{E^*}(\hat{V}^*,\hat{E}^*)$ is part of such a component, its vertices represent linear functions with common slope, say, a. Thus for every pair $(\phi,\psi)\in E^*$,

$$(\text{slope }\phi)\cdot(\text{slope }\psi)=a.$$

Hence all the ϕ, as well as the ψ of the component determined by E^* are parallel. ∎

Now we state (a statistical version of the dual of) a beautiful but almost forgotten result of Beck (see Theorem 3.1 in [2]).

Proposition 12. **(Beck)** *Let $\Phi,\Psi\subset\mathcal{L}$ with $|\Phi|,|\Psi|\le n$ and $E\subset\Phi\times\Psi$ with $|E|\ge cn^2$. Moreover, consider the (not necessarily distinct) intersection points $\Phi\bigcap_E\Psi$. Then at least one of the following two assertions holds (perhaps both):*

(a) *some $c'n^2$ of these intersections coincide; or*

(b) *some $c'n^2$ are all distinct,*

for a $c'=c'(c)$, independent of n.

The proof of the *dual* of this sharper version — even for arbitrary, i.e., non-bipartite graphs E — proceeds on the same lines as Beck's original proof, double-counting the *pairs* from E incident upon each line and using a theorem of Szemerédi and Trotter (Theorem 2 in [18]) in place of Beck's main tool Theorem 1.5. ∎

Remark 13.

1. Case (a) above implies that at least $c'n$ of the Φ and $c'n$ of the Ψ pass through one common point.
2. The statement is false for *finite* planes. Actually, Beck's Theorem is one of the first combinatorial distinctions between Euclidean and finite geometries.

Proof of Lemma 2. Apply Proposition 12 for Φ and Ψ^{-1}. Case (a) — together with the above Remark — implies case (ii) of Lemma 2. Otherwise use Lemma 11. ∎

3. Image sets and concurrency

In this section we study image sets and prove a theorem (Theorem 2) which, while it may be of some interest also on its own right, will play a key role in proving Theorem 1.

3.1. Image sets

Definition 14. For $H \subset \mathbb{R}$ and $\Phi \subset \mathscr{L}$, we put

$$\Phi(H) = \{\phi(h) \ ; \ \phi \in \Phi, h \in H\}$$

and call it an *image set.*

Similarly, the *statistical image set* defined by Φ, H and $E \subset \Phi \times H$ is

$$\Phi_E(H) = \{\phi(h) \ ; \ (\phi, h) \in E\}.$$

Theorem 2. *If $|\Phi|, |H|, |\Phi_E(H)| \leq Cn$ for some $E \subset \Phi \times H$ with $|E| \geq n^2$ then there exists a $\Phi^* \subset \Phi$ which consists of either all parallel or all concurrent lines and $|\Phi^*| \geq c^* n$.*

The proof of this Theorem consists of two steps. First we give a purely graph-theoretic lemma (Lemma 17) to start with. Using that, another lemma (Lemma 19) is proven, which reduces the problem on image sets to one on composition sets.

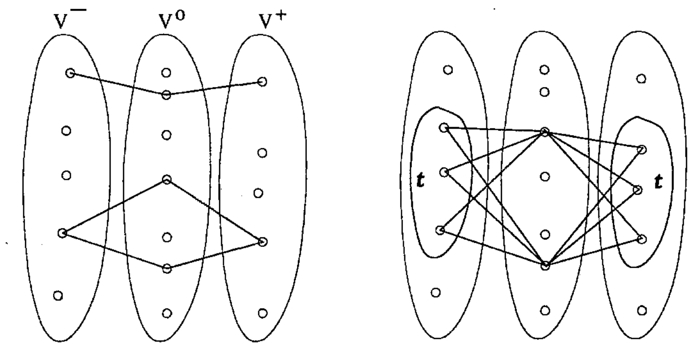
3.2. The Graph Lemma.

Definition 15. An undirected graph $G(V^-, V^0, V^+, E^-, E^+)$ is *double-bipartite* if its vertices consist of three classes V^-, V^0 and V^+ while each edge has one endpoint in V^0 and one in either V^- or V^+. The corresponding edge-sets are E^- and E^+, respectively. The degree of $v_i \in V^0$ in E^- resp. E^+ will be denoted by by $d(v_i^-)$ resp. $d(v_i^+)$.

Definition 16. In an arbitrary graph two vertices are called *t–neighborly,* if they have at least t common neighbors. Similarly, in a double-bipartite graph, two vertices of V^0 are *double-t-neighborly,* if they are t-neighborly both in E^- and E^+.

We shall show that in a double-bipartite graph with many edges, many pairs of the vertices of V^0 are double-highly-neighborly.

The following statement was born with Szemerédi's Regularity Lemma [17] in our mind and, as pointed out by V.T. Sós [19], it can really be proven using that Lemma. However, we present another proof here, one that uses the so-called "C_4–techniques".



a. A double-bipartite graph with a 2-path and a C_4.
b. A double-t-neighborly pair.

Lemma 17. (Graph Lemma) *For every $c, C > 0$ there is a $c^* = c^*(c, C) > 0$ with the following property. Let $G(V^-, V^0, V^+, E^-, E^+)$ be a double-bipartite graph with not more than CN vertices in each class. Assume that G satisfies the following two requirements:*

(i) $d(v_i^+) = d(v_i^-)$ *for each* $v_i \in V^0$;

(ii) $|E^-| = |E^+| \geq cN^2$.

Then there exist c^*N^2 double–c^*N–neighborly pairs in V^0.

Remark 18. Note that (ii) — without (i) — is insufficient even for forcing one single double-1-neighborly pair; e.g. V^- and V^+ might lie in separate connected components of G.

Proof. We shall double–count the number of the C_4-s (i.e. four–cycles) with two "opposite" vertices in V^0 and one more in each of V^- and V^+.

Throughout the proof we shall use the following simple facts for $T \geq 0$:

$$\text{(2)} \qquad \begin{aligned} &\text{if } \sum_1^k t \geq T \text{ then } \sum_1^k t^2 \geq k\,(T/k)^2 = T^2/k; \text{ and similarly,} \\ &\text{if } \sum_1^k t \geq T \text{ then } \sum_1^k \binom{t}{2} \geq k \binom{T/k}{2} = \frac{T}{2}\left(\frac{T}{k} - 1\right). \end{aligned}$$

To start with, we count the number of 2-paths from V^- through V^0 to V^+.

$$\text{(3)} \qquad \#\ \text{2-paths} = \sum_{v_i \in V^0} d^2(v_i) \geq \left(cN^2\right)^2 / CN = \frac{c^2}{C} N^3,$$

by (i), (ii) and (2), where $d(v_i)$ is the (common) value of $d(v_i^+) = d(v_i^-)$. Every C_4 of the required type consists of two 2-paths. Denote the number of such paths from

$v_j^- \in V^-$ to $v_k^+ \in V^+$ by p_{jk}. Of course, $\sum p_{jk} = \#$ 2-paths $\geq \frac{c^2}{C} N^3$ by (3). Hence the number of four–cycles to be counted is

$$\#C_4 = \sum_{j,k} \binom{p_{jk}}{2} \geq \frac{c^4}{3C^4} N^4$$

by (3) and (2) if N is large. Thus $\#C_4 > c' N^4$ for *all* N and a suitable $c' = c'(c, C) > 0$. Now if $c^* = \frac{c'}{C^2(1+C)}$, then at least $c^* N^2$ pairs of V^0 must be double–c^*N–neighborly. ∎

3.3. The reduction lemma

The following assertion forms a sort of link between image sets and composition sets.

Lemma 19. (Reduction Lemma) *If $H \subset \mathbb{R}$, $\Phi \subset \mathscr{L}$ and $|\Phi|, |H|, |\Phi_E(H)| \leq Cn$ for some $E \subset \Phi \times H$ with $|E| \geq cn^2$ then there exists an $E^* \subset \Phi \times \Phi^{-1}$ with $|E^*| \geq c^* n^2$ such that*

$$\left| \Phi \circ_{E^*} \Phi^{-1} \cup \Phi \circ_{E^*} \Phi^{-1} \right| \leq C^* n.$$

Proof. Define a double-bipartite graph as follows:

$$V^0 = \Phi, \quad V^- = H, \quad V^+ = \Phi_E(H);$$
$$E^- = E, \quad E^+ = \{(\phi, \phi(h)) \; ; \; (\phi, h) \in E\}$$

Let c^* be the value provided by the Graph Lemma (Lemma 17) and E_0 the set of double–c^*n-neighborly pairs with $|E_0| \geq c^* n^2$. Put $E^* \stackrel{\text{def}}{=} \{(\phi_i, \phi_j^{-1}); (\phi_i, \phi_j) \in E_0\}$. The set of functions $\Phi \circ_{E^*} \Phi^{-1} \cup \Phi \circ_{E^*} \Phi^{-1}$ all contain at least $c^* n$ points of $(H \times H) \cup (\Phi_E(H) \times \Phi_E(H)) \subset \mathbb{R}^2$.

Now the following (sub)lemma finishes the proof.

Lemma 20. *For every $c > 0$ there is a $C^* = C^*(c) > 0$ with the following property. If $\{P_1, P_2, \ldots\} \subset \mathbb{R}^2$ is a set of M^2 or fewer points then at most $C^* M$ straight lines can contain cM or more of the P_i.*

Proof. According to a theorem of Szemerédi and Trotter (Theorem 2 in [18]), for every $c > 0$ there is a $C^* = C^*(c) > 0$ such that of any T points, not more than $C^* \cdot T^2 / t^3$ lines can contain t or more, if $t \geq c\sqrt{T}$. Use this for $T = M^2$, $t = cM$. ∎

This completes the proof of Lemma 19. ∎

3.4. Proof of Theorem 1

Use the Reduction Lemma (Lemma 19) and then the Main Lemma (Lemma 2), for $\Psi = \Phi^{-1}$ and $E = E^*$.

4. Proof of Theorem 1

Actually, we are going to prove a more general statement, i.e., one under a statistical assumption.

Theorem 3. *For every $C > 0$ there is a $c^* = c^*(C) > 0$ with the following property. Assume that $n \le |\Phi|, |\Psi| \le Cn$ and $|\Phi \circ_E \Psi| \le Cn$ for some $\Phi, \Psi \subset \mathscr{L}$ and $E \subset \Phi \times \Psi$ with $|E| \ge cn^2$.*
Then Φ and Ψ contain some $\Phi^* \subset \Phi$ and $\Psi^* \subset \Psi$ with

$$|(\Phi^* \times \Psi^*) \cap E| \ge c^* n^2,$$

for which
(i) *either both Φ^* and Ψ^* consist of parallel bundles;*
(ii) *or both Φ^* and Ψ^* consist of concurrent bundles.*

Proof. Our goal is to reduce this asymmetric problem to the already solved symmetric case (Lemma 2). In order to do so, we use image sets and Theorem 2.

1. Without loss of generality, assume that the $\phi \in \Phi$, as well as the $\psi \in \Psi$, all have degree at least $cn/(4C)$ in the graph $(\Phi \cup \Psi, E)$. (Otherwise keep on deleting the "trash"; the choice of the constant $c/(4C)$ makes it sure that you cannot throw out everything.)
2. Pick a $u \in \mathbb{R}$ which is "general with respect to Φ", i.e., the vertical line $x = u$ does not pass through the intersection of any pair $\phi_i, \phi_j \in \Phi$. (In other words, the images of u under Φ are all distinct.) Put
$$H = \Phi(\{u\}).$$
3. Define $E' = \{(\psi, \phi(u))\ ;\ (\phi, \psi) \in E\}$, which is a subset of $\Psi \times H$. As the $\phi(u)$ are all distinct, $|E| = |E'| \ge cn^2$.
4. Observe that $|\Psi_{E'}(H)| \le |\Psi \circ_E \Phi| \le Cn$.
5. Use Theorem 2 to get a concurrent or parallel $\Psi^* \subset \Psi$ with $|\Psi^*| \ge c^* n$.
6. By step 1, each $\psi \in \Psi^*$ has degree $c'n$ or more in $(\Phi \cup \Psi, E)$, i.e., the graph generated by Φ and Ψ^* has at least $c'c^*n^2 = c''n^2$ edges. Again, assume without loss of generality that also the $\phi \in \Phi$ have degree $c''n/(4C) = c'''n$ or more in the edge set $(\Phi \times \Psi^*) \cap E$.
7. Repeat steps 2–5 for $\Psi^{*-1} \circ \Phi^{-1} = (\Phi \circ \Psi^*)^{-1}$ and get a $\Phi^{*-1} \subset \Phi^{-1}$ i.e., a $\Phi^* \subset \Phi$ with $|(\Phi^* \times \Psi^*) \cap E| \ge c^{**} n^2$.

8. We are left to show that it is impossible to have such a concurrent Φ^* together with a parallel Ψ^* or vice versa.

Proposition 21. *Assume that the* $\phi \in \Phi$ *all pass through the point* (u,v) *while the* $\psi \in \Psi$ *are all parallel of common slope, say,* $s \neq 0$. *Then*

$$|\Psi \circ_E \Phi| = |E|,$$

i.e., the compositions are all distinct.

Proof. The situation can be thought of as a two–person game. The two players agree upon some ϕ_i of type

$$\phi_i \ : \ x \mapsto s_i(x-u)+v$$

and some ψ_j of type

$$\psi_j \ : \ x \mapsto sx + t_j.$$

To start with, the first person picks a pair (i,j) and tells the other one the coefficients of the composition

$$\psi_j \circ \phi_i \ : \ x \mapsto s \cdot s_i(x-u) + sv + t_j.$$

Now the second player must (and can) find out i and j.

Indeed, it is easy to first determine i (via s_i) from the leading coefficient. After having done so, also t_j and thus j can be found. ∎

9. This completes the proof of Theorem 3 and that of Theorem 1, as well. ∎

Acknowledgments. The author wishes to thank V. T. Sós for all her help and encouragements. Moreover, the comments and ideas of Z. Király, L. A. Székely, T. Szőnyi and M. Simonovits were very useful; not even the notion of "commutator pairs" would have occurred to us without some remarks of Simonovits. Finally, we are grateful to E. Makai for reading the first version of the manuscript very carefully.

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